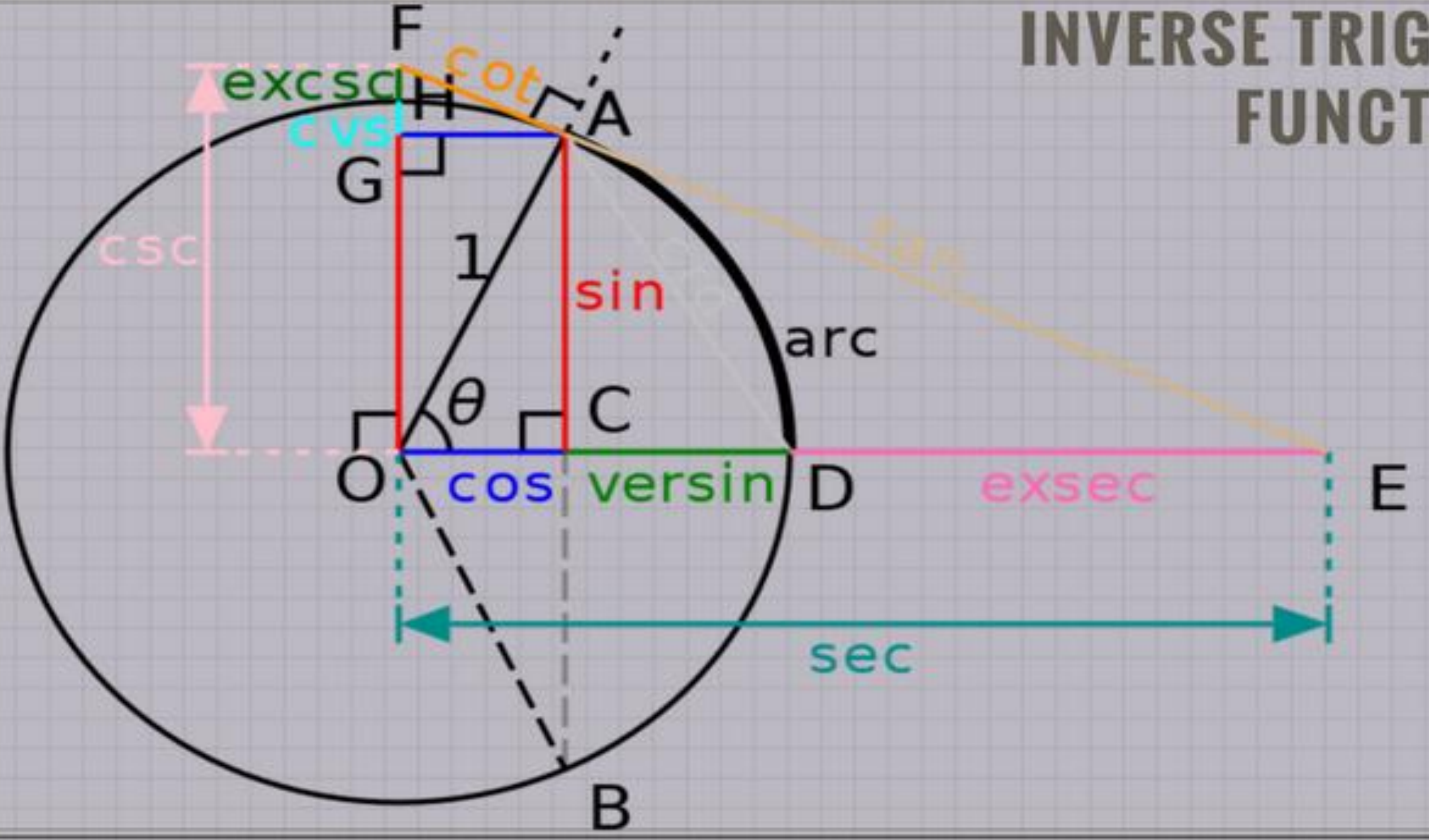
INVERSE TRIGONOMETRIC FUNCTIONS



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MODULE 1

DOMAIN & PRINCIPAL VALUE BRANCHES

Functions	Domain	Range (Principal Value Branches)
$y = \sin^{-1} x$	$[-1,1]$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
$y = \cos^{-1} x$	$[-1,1]$	$[0, \pi]$
$y = \text{cosec}^{-1} x$	$R - (-1,1)$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
$y = \sec^{-1} x$	$R - (-1, 1)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
$y = \tan^{-1} x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$y = \cot^{-1} x$	R	$(0, \pi)$

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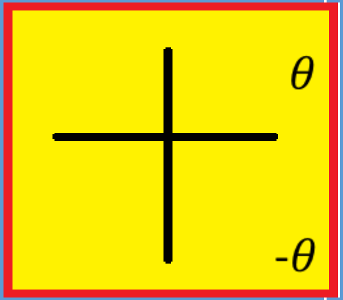
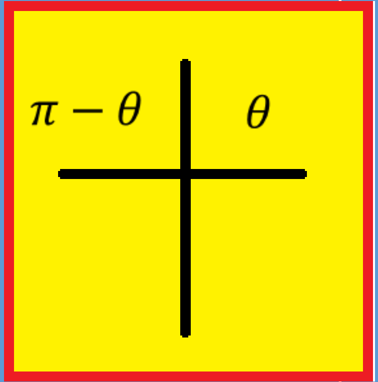
$y = \sin^{-1} x$



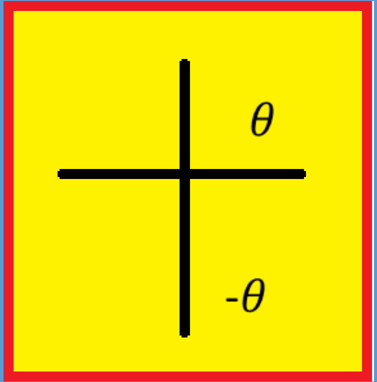
$[-1,1] \rightarrow \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

$y = \cos^{-1} x$



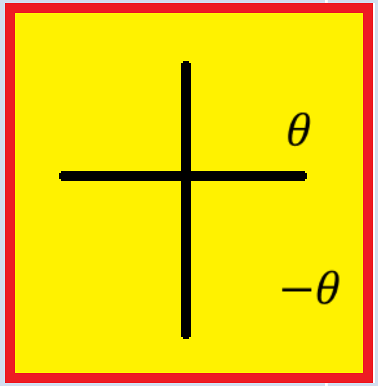
$[-1,1] \rightarrow [0, \pi]$

$Y = \tan^{-1} x$



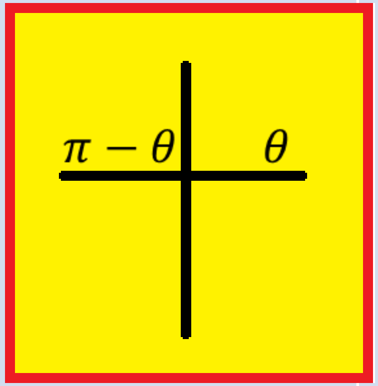
$R \longrightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$Y = \text{cosec}^{-1} x$



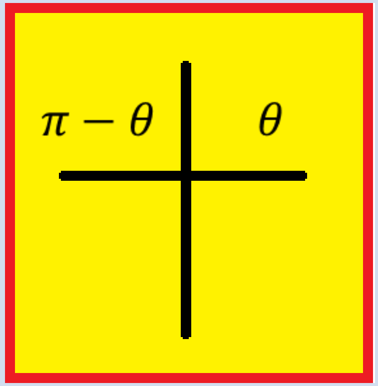
$R - (-1,1) \longrightarrow \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

$Y = \sec^{-1} x$



$R - (-1,1) \longrightarrow [0, \pi] - \{\frac{\pi}{2}\}$

$Y = \cot^{-1} x$



$R \longrightarrow (0, \pi)$

Find the principle values of the following

$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

1) $sin^{-1}\left(\frac{1}{2}\right)$

Let x = $sin^{-1}\left(\frac{1}{2}\right)$

sin x = $\frac{1}{2}$

θ

$\therefore x = \frac{\pi}{6}$

2) $sin^{-1}\left(-\frac{1}{2}\right)$

$[-1,1] \to \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

Let x = $sin^{-1}\left(\frac{-1}{2}\right)$

sin x = $\frac{-1}{2}$

$\therefore x = \frac{-\pi}{6}$

$-\theta$

Type equation here.

3) $cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$[-1,1] \to [0, \pi]$

Let x = $cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$cos\, x = \frac{\sqrt{3}}{2}$

θ

$\therefore x = \frac{\pi}{6}$

4) $cos^{-1}\left(-\frac{1}{2}\right)$

Let x = $cos^{-1}\left(\frac{-1}{2}\right)$

cos x = $\frac{-1}{2}$

$\pi - \theta$

$[-1,1] \to [0, \pi]$

x = $\pi - \theta = \pi - \frac{\pi}{3}$

5) $cosec^{-1}(-2)$

$R - (-1,1) \to$

$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

cosec x = -2

$-\theta$

x = $\frac{-\pi}{6}$

6) $sec^{-1}(-\sqrt{2})$

x = $sec^{-1}(-\sqrt{2})$

sec x = $-\sqrt{2}$

$\pi - \theta$

x = $\pi - \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$\frac{\pi}{4}$

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$7) \; cosec^{-1}(2)$ (QUESTION FOR POLL)

A) $\frac{\pi}{6}$ B) $\frac{2\pi}{3}$ C) $\frac{\pi}{3}$ D) $\frac{\pi}{4}$ E) $\frac{5\pi}{6}$

8) $\tan^{-1}(1)$ (QUESTION FOR POLL)

A) $\frac{\pi}{2}$ B) $\frac{\pi}{4}$ C)) $\frac{\pi}{3}$ D) $\frac{3\pi}{4}$ E) NONE

9) $cos^{-1}\left(-\frac{1}{2}\right)$

A) - $\frac{\pi}{3}$ B) $\frac{\pi}{6}$ C) $\frac{5\pi}{6}$ D) $\frac{2\pi}{3}$ E) NONE

10) $sin^{-1}\left(\frac{1}{2}\right)$

A) $\frac{\pi}{6}$ B) $\frac{2\pi}{3}$ C) $\frac{\pi}{3}$ D) $\frac{\pi}{4}$ E) $\frac{5\pi}{6}$

11) $tan^{-1}(1) + cos^{-1}\left(-\frac{1}{2}\right) + sin^{-1}\left(\frac{1}{2}\right)$

$= \frac{\pi}{4} + (\pi - \frac{\pi}{3}) + \frac{\pi}{6}$

$= \frac{13\pi}{12}$

12) $\tan^{-1}(-1)$

A) $\frac{\pi}{2}$ B) - $\frac{\pi}{4}$ C)) $\frac{\pi}{3}$ D) $\frac{3\pi}{4}$ E) NONE

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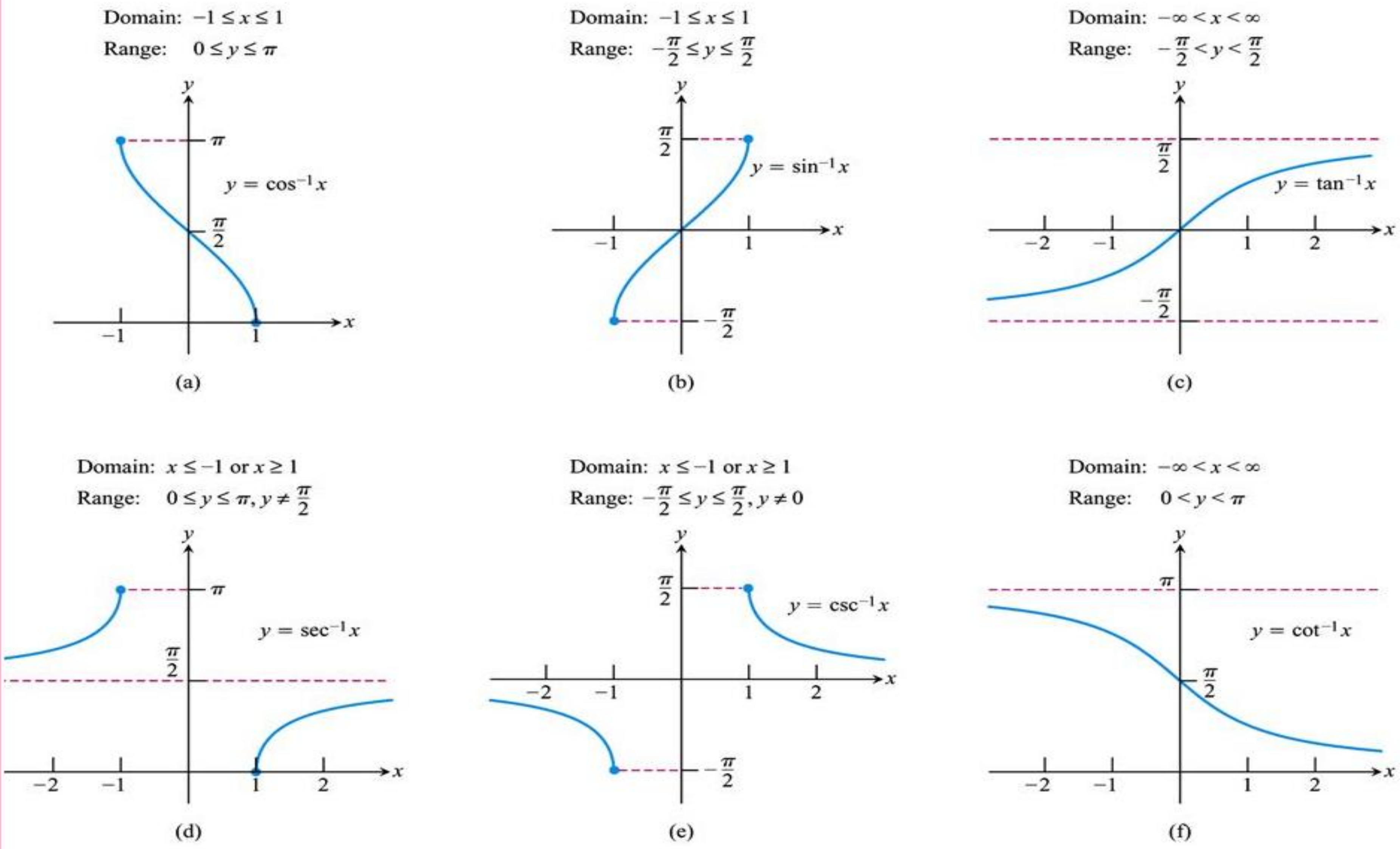

Domain: $-1 \le x \le 1$
Range: $0 \le y \le \pi$
$y = \cos^{-1} x$
(a)
Domain: $-1 \le x \le 1$
Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \sin^{-1} x$
(b)
Domain: $-\infty < x < \infty$
Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$
$y = \tan^{-1} x$
(c)
Domain: $x \le -1$ or $x \ge 1$
Range: $0 \le y \le \pi, y \ne \frac{\pi}{2}$
$y = \sec^{-1} x$
(d)
Domain: $x \le -1$ or $x \ge 1$
Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}, y \ne 0$
$y = \csc^{-1} x$
(e)
Domain: $-\infty < x < \infty$
Range: $0 < y < \pi$
$y = \cot^{-1} x$
(f)

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HOME WORK – EX 2.1

Q . 8 , 9 , 12 , 13 , 14.

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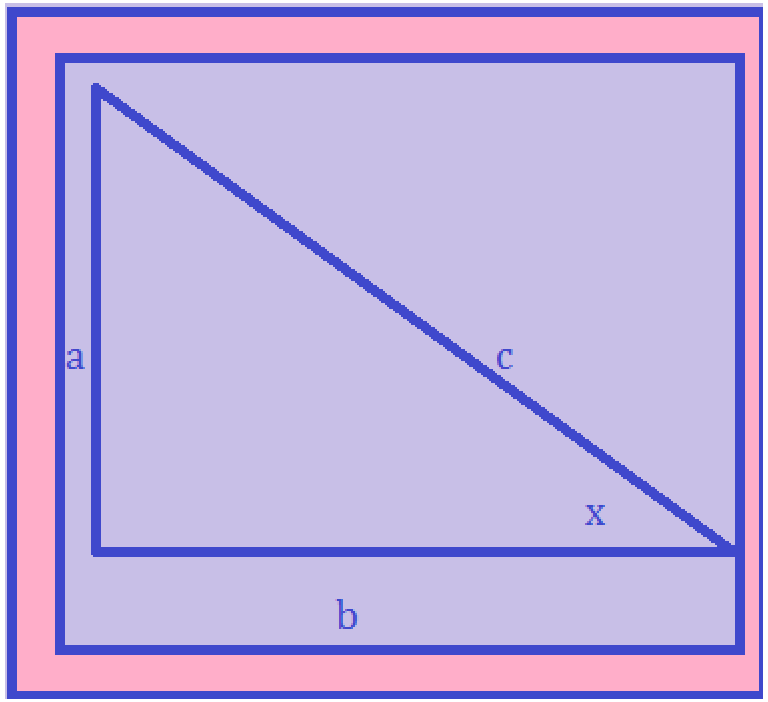
INVERSE TRIGONOMETRIC FUNCTIONS- MODULE 2



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$\sin^{-1}(-x) = -\sin^{-1}x,\ x \in [-1, 1]$

$\tan^{-1}(-x) = -\tan^{-1}x,\ x \in R$

$\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x,\ |x| \geq 1$

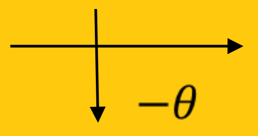
$\cos^{-1}(-x) = \pi - \cos^{-1}x,\ x \in [-1, 1]$

$\sec^{-1}(-x) = \pi - \sec^{-1}x,\ |x| \geq 1$

$\cot^{-1}(-x) = \pi - \cot^{-1}x,\ x \in R$

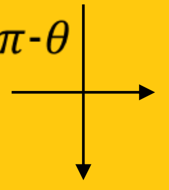
$\sin^{-1}\left(-\frac{1}{2}\right) = -\sin^{-1}(1/2)$

$= \frac{-\pi}{6}$



$\cos^{-1}\left(-\frac{1}{2}\right) = \pi - \cos^{-1}\frac{1}{2} =$

$= \pi - \frac{\pi}{3} = \frac{2\pi}{3}$



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PROPERTIES OF INVERSE TRIGO.FUNCTIONS

$\sin^{-1}(\sin\theta) = \theta$, $\sin(\sin^{-1} x) = x$

$\cos^{-1}(\cos\theta) = \theta$, $\cos(\cos^{-1} x) = x$

$\tan^{-1}(\tan\theta) = \theta$, $\tan(\tan^{-1} x) = x$

$\cot^{-1}(\cot\theta) = \theta$, $\cot(\cot^{-1} x) = x$

$\sec^{-1}(\sec\theta) = \theta$, $\sec(\sec^{-1} x) = x$

$\mathrm{cosec}^{-1}(\mathrm{cosec}\,\theta) = \theta$, $\mathrm{cosec}(\mathrm{cosec}^{-1} x) = x$

$f^{-1}(f(x)) = x$

$\sin^{-1} x = \mathrm{cosec}^{-1}(1/x)$

$\cos^{-1} x = \sec^{-1}(1/x)$

$\tan^{-1} x = \cot^{-1}(1/x)$

$\sin\left(\frac{\pi}{6}\right) = ½$: $\mathrm{cosec}\left(\frac{\pi}{6}\right) = 2$

$\sin^{-1} 1/2 = \frac{\pi}{6}$: $\mathrm{cosec}^{-1} 2 = \frac{\pi}{6}$

$\therefore \sin^{-1} 1/2 = \mathrm{cosec}^{-1} 2$

$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$

$\sec^{-1} x + \mathrm{cosec}^{-1} x = \frac{\pi}{2}$

$\sin^{-1} 1/2 + \cos^{-1} 1/2 = \frac{\pi}{6} + \frac{\pi}{3}$

$= \frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{2}$

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properties continue.....

(i) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right); xy < 1$

ii) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right); xy > -1$

(i) $2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right); |x| \le 1$

(ii) $2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right); x \ge 0$

(iii) $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right); -1 < x < 1$

tan(A+B) = $\frac{\tan A+\tan B}{1-tanA.tanB}$

Sin 2x = $\frac{2tanx}{1+tan^2x}$

Cos2x = $\frac{1-tan^2x}{1+tan^2x}$

Tan2x = $\frac{2tanx}{1-tan^2x}$

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Each property is a formula..There are hundreds of questions hidden in a formula

$$2\tan^{-1}x=\cos^{-1}\frac{1-x^2}{1+x^2}$$

HOW???

We know, $\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-x^2}{1+x^2}$

$$2\theta = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

Both sides are balanced for $x \geq 0$

Similarly, $\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2x}{1-x^2}$

$$2\theta = 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}$$

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RECOLLECTING RANGE

$\sin^{-1}\sin(x) = x \; if \; x \; is \; in \; its \; domain$

1)$\sin^{-1}\sin(\frac{\pi}{6}) = \frac{\pi}{6}$

2) $\sin^{-1}\sin(\frac{5\pi}{6}) = \sin^{-1}\sin(\pi - \frac{\pi}{6}) = \sin^{-1}\sin(\frac{\pi}{6}) = \frac{\pi}{6}$

3) $\sin^{-1}\sin(-\frac{\pi}{6}) = -\frac{\pi}{6}$

4)$\cos^{-1}\cos\frac{2\pi}{3} = 2\pi/3$

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QUESTIONS FOR POLL

Think 2 minutes

1) $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

A) $\frac{2\pi}{3}$ B) $\frac{\pi}{3}$ C) $\frac{-\pi}{3}$ D) NONE

2) $\tan^{-1}(-1) + \cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

A) $\frac{\pi}{3}$ B) $\frac{\pi}{6}$ *C)* $\frac{2\pi}{3}$ *D)* $\frac{\pi}{4}$ E) $\frac{5\pi}{6}$

3) $\tan^{-1}\left(tan\frac{3\pi}{4}\right)$

A) $\frac{\pi}{4}$ B) $\frac{3\pi}{4}$ C) $\frac{5\pi}{4}$ D) -$\frac{\pi}{4}$ ***E) NONE***

4) $\cos^{-1}\cos\left(\frac{7\pi}{6}\right)$

A) $\frac{\pi}{3}$ B) $\frac{\pi}{6}$ *C)* $\frac{2\pi}{3}$ *D)* $\frac{\pi}{4}$ E) $\frac{5\pi}{6}$

5) $\sin^{-1}\left(\sin\frac{3\pi}{5}\right)$

A) $\frac{3\pi}{5}$ B) $\frac{2\pi}{5}$ C) $\frac{\pi}{5}$ D) -$\frac{\pi}{5}$ E) NONE

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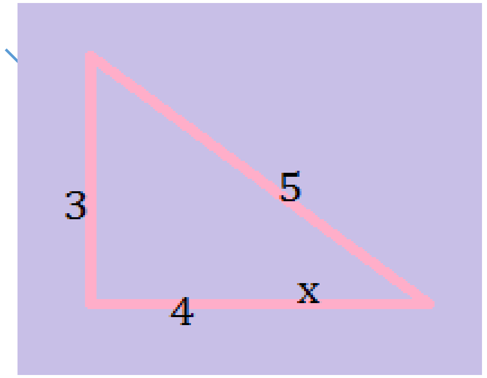
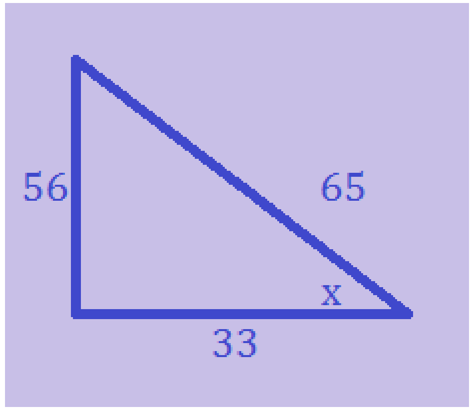
Pythagoras again……Here with Inverse

$\sin^{-1}\left(\frac{3}{5}\right) = \cos^{-1}(? -)$

$\cos^{-1}\left(\frac{33}{65}\right) = \tan^{-1}(-)$

$\tan^{-1}\left(\frac{24}{7}\right) = \sin^{-1}(\quad)$---- =- $\cos^{-1}(\quad)$------





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HOME WORK

- 1) $cos^{-1}\left(\frac{1}{2}\right) + 2sin^{-1}\left(\frac{1}{2}\right)$
- 2) $tan^{-1}(\sqrt{3}) + sec^{-1}(-2) + cosec^{-1}\left(\frac{2}{\sqrt{3}}\right)$
- 3) $sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) + cos^{-1}\left(\frac{1}{2}\right) + 2cot^{-1}(-1)$
- 4) $tan^{-1}\left(tan(\frac{2\pi}{3})\right)$
- 5) $sin^{-1}(sin\frac{5\pi}{4})$
- 6)Write a branch of sin^{-1}x other than principal value branch

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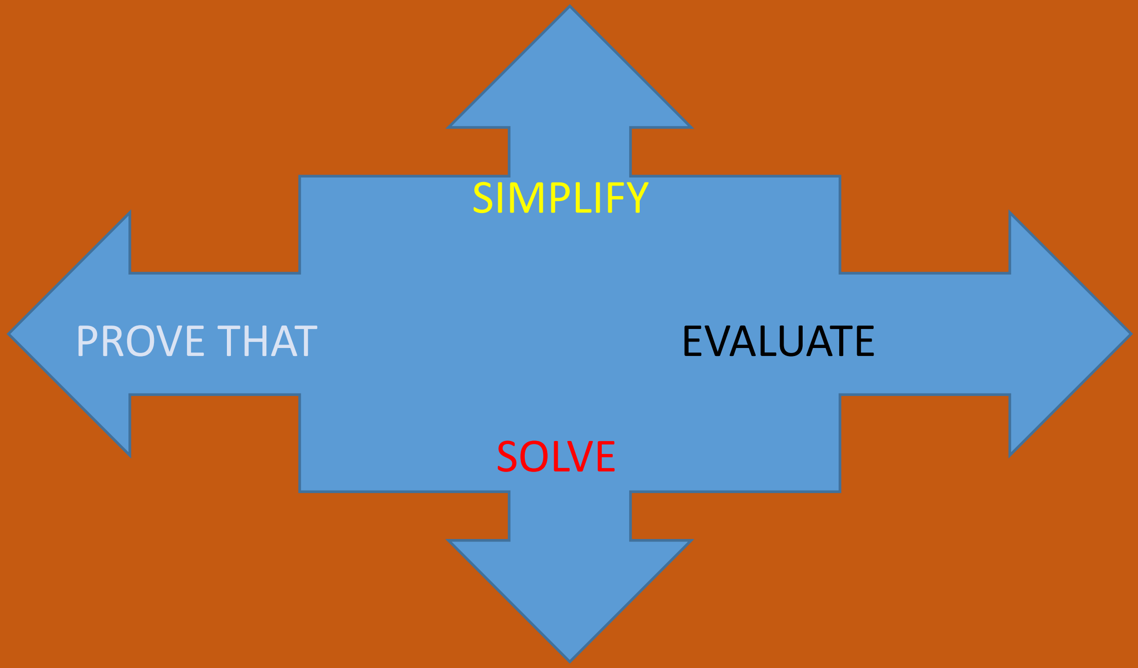
INVERSE TRIGONOMETRY-MODULE 3

TOPICS

- PROBLEM SOLVING



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HOW ???????

DIFFERENT METHODS

INVERSE TRIGO.PROPERTIES

SUBSTITUTION USING TRIGO.IDENTITIES

TRIANGLE METHOD

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PROVE THE FOLLOWING

Using properties

P.T $\tan^{-1}(1/7) + \tan^{-1}(1/13) = \tan^{-1}(2/9)$

Consider LHS

$\tan^{-1}(\frac{1}{7}) + \tan^{-1}(\frac{1}{13})$

We know that, Formula

$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$

property

According to the formula, we can write as

$= \tan^{-1}\left(\frac{\frac{1}{7}+\frac{1}{13}}{1-\frac{1}{7}\times\frac{1}{13}}\right)$

$= \tan^{-1}\left(\frac{\frac{13+7}{91}}{\frac{91-1}{91}}\right)$

$= \tan^{-1}\left(\frac{20}{90}\right)$

$= \tan^{-1}\left(\frac{2}{9}\right)$

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PROVE ....

Use PROPERTIES

Question

Prove that $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}$

Solution Using $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$$\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{5}\right) = \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{5}}{1-\frac{1}{2}\cdot\frac{1}{5}}\right) = \tan^{-1}\left(\frac{\frac{7}{10}}{\frac{9}{10}}\right) = \tan^{-1}\left(\frac{7}{9}\right)$$

$$\therefore \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \tan^{-1}\left(\frac{7}{9}\right) + \tan^{-1}\left(\frac{1}{8}\right)$$

$$= \tan^{-1}\left(\frac{\frac{7}{9}+\frac{1}{8}}{1-\frac{7}{9}\cdot\frac{1}{8}}\right) = \tan^{-1}\left(\frac{\frac{65}{72}}{\frac{65}{72}}\right)$$

$$= \tan^{-1}(1) = \tan^{-1}\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4}$$

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Again Pythagoras………

Triangle method

Question 7:

Prove $\tan^{-1}\frac{63}{16} = \sin^{-1}\frac{5}{13} + \cos^{-1}\frac{3}{5}$

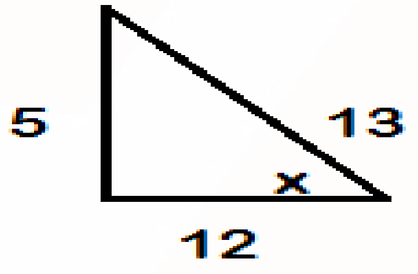
Solution 7:

Let $\sin^{-1}\frac{5}{13} = x$

Then, $\sin x = \frac{5}{13} \Rightarrow \cos x = \frac{12}{13}$.

$\therefore \tan x = \frac{5}{12} \Rightarrow x = \tan^{-1}\frac{5}{12}$

$\therefore \sin^{-1}\frac{5}{13} = \tan^{-1}\frac{5}{12}$ .....(1)

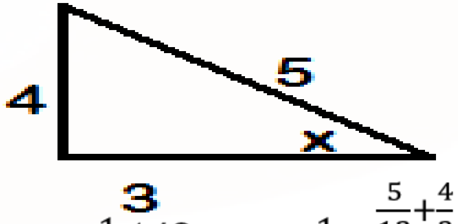


Let $\cos^{-1}\frac{3}{5} = y$.

Then, $\cos y = \frac{3}{5} \Rightarrow \sin y = \frac{4}{5}$.

Thus, $\tan y = \frac{4}{3} \Rightarrow y = \tan^{-1}\frac{4}{3}$

$\therefore \cos^{-1}\frac{3}{5} = \tan^{-1}\frac{4}{3}$ .....(2)



Using (1) and (2), we have

$\tan^{-1}\frac{5}{12} + \tan^{-1}4/3 = \tan^{-1}\frac{\frac{5}{12}+\frac{4}{3}}{1-\frac{5}{12}\times\frac{4}{3}}$.

$= \tan^{-1}63/16$

R.H.S.

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PROVE THAT.............

Prove $\sin^{-1}\frac{8}{17}+\sin^{-1}\frac{3}{5}=\tan^{-1}\frac{77}{36}$

Answer

Let $\sin^{-1}\frac{8}{17}=x$. Then, $\sin x=\frac{8}{17}\Rightarrow \cos x=\sqrt{1-\left(\frac{8}{17}\right)^2}=\sqrt{\frac{225}{289}}=\frac{15}{17}$.

$\therefore \tan x=\frac{8}{15}\Rightarrow x=\tan^{-1}\frac{8}{15}$

$\therefore \sin^{-1}\frac{8}{17}=\tan^{-1}\frac{8}{15}$ ...(1)

Now, let $\sin^{-1}\frac{3}{5}=y$. Then, $\sin y=\frac{3}{5}\Rightarrow \cos y=\sqrt{1-\left(\frac{3}{5}\right)^2}=\sqrt{\frac{16}{25}}=\frac{4}{5}$.

$\therefore \tan y=\frac{3}{4}\Rightarrow y=\tan^{-1}\frac{3}{4}$

$\therefore \sin^{-1}\frac{3}{5}=\tan^{-1}\frac{3}{4}$ ...(2)

Now, we have:

$$\text{L.H.S.}=\sin^{-1}\frac{8}{17}+\sin^{-1}\frac{3}{5}$$

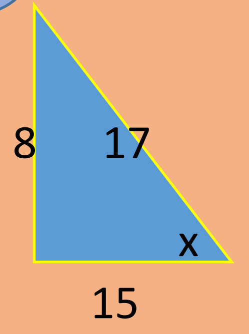
$$=\tan^{-1}\frac{8}{15}+\tan^{-1}\frac{3}{4} \quad \text{[Using (1) and (2)]}$$

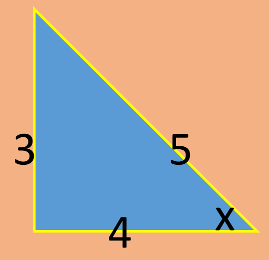
$$=\tan^{-1}\frac{\frac{8}{15}+\frac{3}{4}}{1-\frac{8}{15}\times\frac{3}{4}}$$

$$=\tan^{-1}\left(\frac{32+45}{60-24}\right) \quad \left[\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}\right]$$

$$=\tan^{-1}\frac{77}{36}=\text{R.H.S.}$$

TRIANGLE METHOD





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SIMPLIFY…..

Simplify: $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$;

TRIGONOMETRIC IDENTITIES

Consider $\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}$

$$= \frac{\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)^2}{\left(\sqrt{1+\sin x}\right)^2-\left(\sqrt{1-\sin x}\right)^2} \quad \text{(by rationalizing)}$$

$$= \frac{(1+\sin x)+(1-\sin x)+2\sqrt{(1+\sin x)(1-\sin x)}}{1+\sin x-1+\sin x}$$

$$= \frac{2\left(1+\sqrt{1-\sin^2 x}\right)}{2\sin x} = \frac{1+\cos x}{\sin x} = \frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$= \cot\frac{x}{2}$$

$$\therefore \text{L.H.S.} = \cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \cot^{-1}\left(\cot\frac{x}{2}\right) = \frac{x}{2} = \text{R.H.S.}$$

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SIMPLIFY……………….

$$\tan^{-1}\frac{\sqrt{1+x^2}-1}{x},\ x\neq 0$$

Answer

$$\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$$

Put $x=\tan\theta \Rightarrow \theta=\tan^{-1}x$

$$\therefore \tan^{-1}\frac{\sqrt{1+x^2}-1}{x}=\tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$=\tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)=\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$=\tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$$

$$=\tan^{-1}\left(\tan\frac{\theta}{2}\right)=\frac{\theta}{2}=\frac{1}{2}\tan^{-1}x$$

How we will simplify Using substitution

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SOLVE............

If $\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$, then find the value of x.

Answer

$$\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)}\right]=\frac{\pi}{4}$$

$$\left[\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}\right]$$

PROPERTIES

$$\Rightarrow \tan^{-1}\left[\frac{(x-1)(x+2)+(x+1)(x-2)}{(x+2)(x-2)-(x-1)(x+1)}\right]=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{x^2+x-2+x^2-x-2}{x^2-4-x^2+1}\right]=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{2x^2-4}{-3}\right]=\frac{\pi}{4}$$

$\frac{2x^2-4}{-3}=1$, Then x = $\pm\frac{1}{\sqrt{2}}$

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SIMPLIFY $\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$, $0 < x < \pi$

$$\tan^{-1}\left(\frac{\cos(x)-\sin(x)}{\cos(x)+\sin(x)}\right), \quad 0 < x < \pi$$

$= \tan^{-1}\left(\frac{1-\tan(x)}{1+\tan(x)}\right)$ diving the numerator and denominator by $\cos x$

$= \tan^{-1}\left(\frac{\tan\frac{\pi}{4}-\tan(x)}{1+\tan\frac{\pi}{4}\tan(x)}\right)$ as $\tan\frac{\pi}{4} = 1$

$= \tan^{-1}\tan(\frac{\pi}{4} - x) = \frac{\pi}{4} - x$

$\tan(\frac{\pi}{4} - x) =$
???

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P.T $\tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right] = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x$

Taking LHS, we get:

$$\tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]$$

let $x = \cos 2\theta$

$$\tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right] = \tan^{-1}\left[\frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right]$$

$$= \tan^{-1}\left[\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right]$$

$$= \tan^{-1}\left[\frac{1-\tan\theta}{1+\tan\theta}\right]$$

$$= \tan^{-1}\tan\left(\frac{\pi}{4}-\theta\right)$$

$$= \left(\frac{\pi}{4}-\theta\right)$$

$$= \frac{\pi}{4}-\theta$$

$$= \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x$$

SUBSTITUTION

X = COS2θ

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PROVE THAT

$$\tan^{-1}\left(\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right)$$

$$=\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2$$

We have $\tan^{-1}\left[\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right]$

Let $\quad x^2=\cos 2\theta \Rightarrow \theta=\frac{1}{2}\cos^{-1}x^2$

$$\therefore \quad \text{L.H.S.}=\tan^{-1}\left[\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right]$$

$$=\tan^{-1}\left[\frac{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}\right]$$

$$=\tan^{-1}\left[\frac{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}\right]$$

$$=\tan^{-1}\left[\frac{1+\tan\theta}{1-\tan\theta}\right]$$

$$=\tan^{-1}\left[\frac{\tan\frac{\pi}{4}+\tan\theta}{1-\tan\frac{\pi}{4}\tan\theta}\right]$$

$$=\tan^{-1}\left(\tan\left(\frac{\pi}{4}+\theta\right)\right)=\frac{\pi}{4}+\theta=\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2$$

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SOLVE.......

$\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}$

Solution We have $\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}$

or $$\tan^{-1}\left(\frac{2x+3x}{1-2x\times 3x}\right) = \frac{\pi}{4}$$

i.e. $$\tan^{-1}\left(\frac{5x}{1-6x^2}\right) = \frac{\pi}{4}$$

Therefore $$\frac{5x}{1-6x^2} = \tan\frac{\pi}{4} = 1$$

or $$6x^2 + 5x - 1 = 0 \text{ i.e., } (6x-1)(x+1) = 0$$

which gives $$x = \frac{1}{6} \text{ or } x = -1.$$

Since $x = -1$ does not satisfy the equation, as the L.H.S. of the equation becomes negative, $x = \frac{1}{6}$ is the only solution of the given equation.

Verification required

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SIMPLIFY

Simplify $\tan^{-1}\left[\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right]$, if $\frac{a}{b}\tan x > -1$

$$\tan^{-1}\left[\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right]$$

We write $\frac{a\cos x - b\sin x}{b\cos x + a\sin x}$ *in form of tan*

We know that

$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

We need denominator in form

1 + tan x tan y

hence we need 1 instead of b cos x

So dividing both numerator and denominator by b cos x

$$= \tan^{-1}\left[\frac{\frac{a\cos x - b\sin x}{b\cos x}}{\frac{b\cos x + a\sin x}{b\cos x}}\right]$$

$$= \tan^{-1}\left[\frac{\frac{a\cos x}{b\cos x} - \frac{b\sin x}{b\cos x}}{\frac{b\cos x}{b\cos x} + \frac{a\sin x}{b\cos x}}\right]$$

$$= \tan^{-1}\left[\frac{\frac{a}{b} - \frac{\sin x}{\cos x}}{1 + \frac{a\sin x}{b\cos x}}\right]$$

$$= \tan^{-1}\left[\frac{\frac{a}{b} - \tan x}{1 + \frac{a}{b}\tan x}\right]$$

Using equation

$$\tan^{-1}\left(\frac{x-y}{1+xy}\right) = \tan^{-1} x - \tan^{-1} y$$

Replacing x with $\frac{a}{b}$ *and y with tan x*

$$= \tan^{-1}\frac{a}{b} - \tan^{-1}(\tan x)$$

$$= \tan^{-1}\frac{a}{b} - x$$

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TOOL → MAY I HELP YOU

S.No.	Expression	Substitute
1.	$\sqrt{a^2 - x^2}$	$x = a\sin\theta$ or $x = a\cos\theta$
2.	$\sqrt{a^2 + x^2}$	$x = a\tan\theta$ or $x = a\cot\theta$
3.	$\sqrt{x^2 - a^2}$	$x = a\sec\theta$ or $x = a\,\mathrm{cosec}\,\theta$
4.	$\sqrt{a + x}$ or $\sqrt{a - x}$	$x = a\cos\theta$ or $x = a\cos 2\theta$

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WRITE IN SIMPLEST FORM

$$\because y = \tan^{-1}\left(\frac{3a^2x - x^3}{a^3 - 3ax^2}\right)$$

$$x = a\tan\theta \qquad \therefore \tan\theta = \frac{x}{a} \qquad \Rightarrow \qquad \theta = \tan^{-1}\left(\frac{x}{a}\right)$$

$$\therefore \qquad y = \tan^{-1}\left(\frac{3a^2 \cdot a\tan\theta - a^3\tan^3\theta}{a^3 - 3a \cdot a^2\tan^2\theta}\right)$$

$$= \tan^{-1}\left(\frac{3a^3\tan\theta - a^3\tan^3\theta}{a^3 - 3a^3\tan^2\theta}\right)$$

$$= \tan^{-1}\left[\frac{a^3(3\tan\theta - \tan^3\theta)}{a^3(1 - 3\tan^2\theta)}\right]$$

$$= \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right)$$

$$= \tan^{-1}\tan(3\theta) \qquad \left[\because \tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}\right]$$

$$= \mathbf{3\theta = 3\tan^{-1}\left(\frac{x}{a}\right)}$$

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HOW TO PROVE BY SUBSTITUTION

(h) (i) $3\sin^{-1}x = \sin^{-1}(3x - 4x^3);\ \frac{-1}{2} \le x \le \frac{1}{2}$

(ii) $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x);\ \frac{1}{2} \le x \le 1$

(iii) $3\tan^{-1}x = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right);\ \frac{-1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

(i) (i) $\sin^{-1}x = \cos^{-1}(\sqrt{1 - x^2}) = \tan^{-1}\left(\frac{x}{\sqrt{1 - x^2}}\right)$

(ii) $\cos^{-1}x = \sin^{-1}(\sqrt{1 - x^2}) = \tan^{-1}\left(\frac{\sqrt{1 - x^2}}{x}\right)$

(iii) $\tan^{-1}x = \sin^{-1}\left(\frac{x}{\sqrt{1 + x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1 + x^2}}\right)$

Substitution

Your answers here---------

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TEXT BOOK EXAMPLE

Example 11 Show that $\sin^{-1}\frac{12}{13} + \cos^{-1}\frac{4}{5} + \tan^{-1}\frac{63}{16} = \pi$

Solution Let $\sin^{-1}\frac{12}{13} = x$, $\cos^{-1}\frac{4}{5} = y$, $\tan^{-1}\frac{63}{16} = z$

Then $\sin x = \frac{12}{13}$, $\cos y = \frac{4}{5}$, $\tan z = \frac{63}{16}$

Therefore $\cos x = \frac{5}{13}$, $\sin y = \frac{3}{5}$, $\tan x = \frac{12}{5}$ and $\tan y = \frac{3}{4}$

We have $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{\frac{12}{5}+\frac{3}{4}}{1-\frac{12}{5}\times\frac{3}{4}} = -\frac{63}{16}$

Hence $\tan(x+y) = -\tan z$

i.e., $\tan(x+y) = \tan(-z)$ or $\tan(x+y) = \tan(\pi - z)$

Therefore $x + y = -z$ or $x + y = \pi - z$

Since x, y and z are positive, $x + y \neq -z$ (Why?)

Hence $x + y + z = \pi$ or $\sin^{-1}\frac{12}{13} + \cos^{-1}\frac{4}{5} + \tan^{-1}\frac{63}{16} = \pi$

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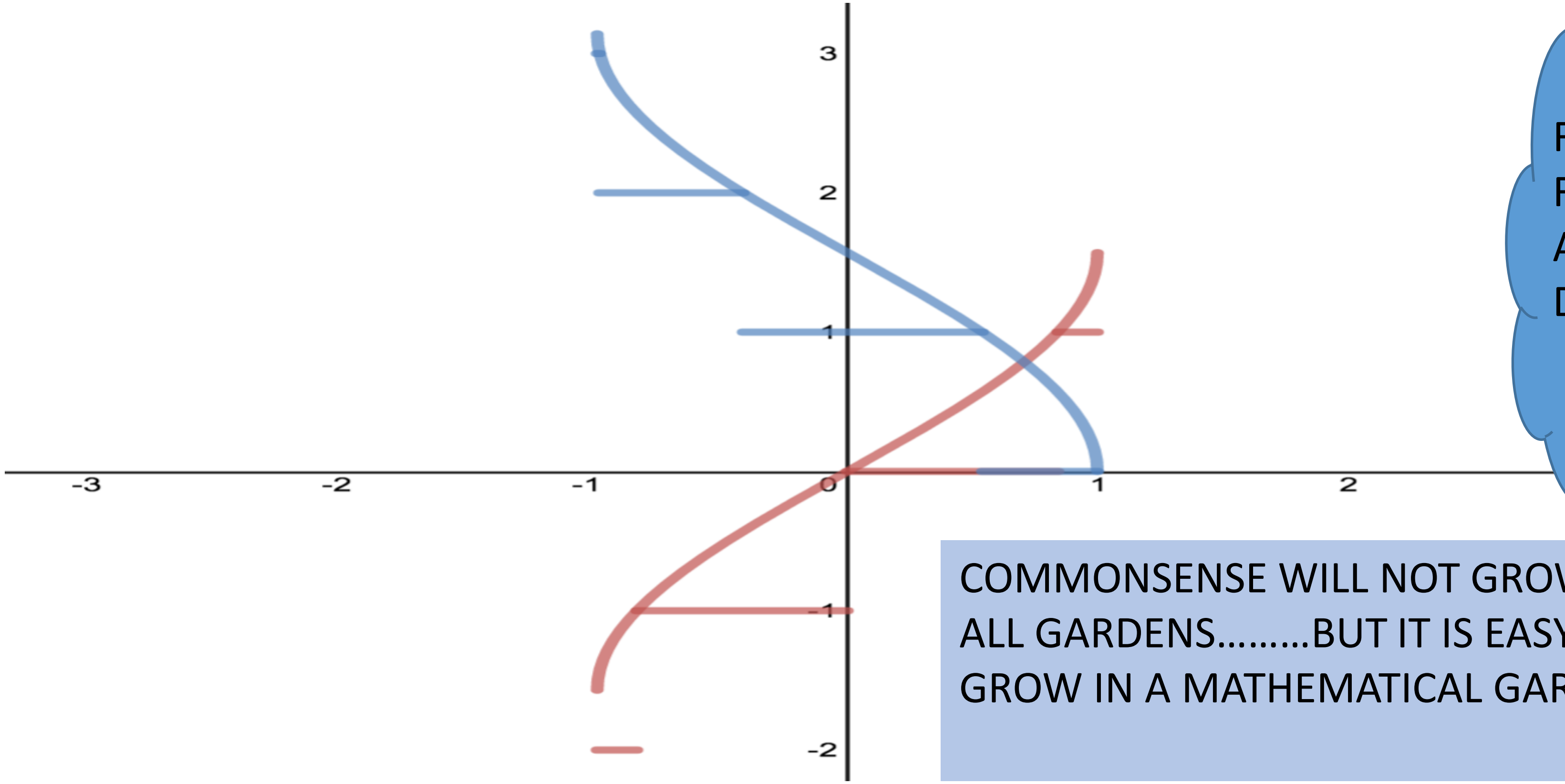
HOME WORK: MIS EX.5,6,7,8,9,12,14,16



FEEL FREE TO ASK ANY DOUBTS

COMMONSENSE WILL NOT GROW IN ALL GARDENS………BUT IT IS EASY TO GROW IN A MATHEMATICAL GARDEN

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